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Cosmic String in Kerr–Newman–Kasuya Spacetime

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We study the equilibrium configurations of a cosmic string described by the Nambu action in curved spacetime such as the Kerr–Newman–Kasuya spacetime, which is the Kerr–Newman spacetime involved with extra magnetic monopole charge. In this study it is interesting to note that the physical results remain the same whether or not the magnetic monopole does exist in nature.

1. INTRODUCTION

Cosmic strings are topologically stable objects which might have been formed during a phase transition in the early universe (Kibble, 1976). The cosmic strings formed during a phase transition in the early universe might provide the seeds needed for galaxy formation (Zeldovich, 1980). Such strings are predicted in certain grand unified theories.

Recently Frolov *et al.* (1989) studied the possible equilibrium configurations of a cosmic string in curved spacetime such as the Kerr–Newman black hole spacetime. In this paper, we would like to extend the results of Frolov *et al.* in the Kerr–Newman–Kasuya spacetime. The Kerr–Newman–Kasuya spacetime is the Kerr–Newman spacetime involved with extra magnetic monopole charge. We give reasons to believe that magnetic monopoles exist on the grounds of the symmetry that they would introduce in the field equation of electromagnetism. This monopole hypothesis was propounded by Dirac relatively long ago. The ingenious suggestion by Dirac that the magnetic monopole does exist was neglected due to the failure to detect such a particle.

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However, in recent years the development of gauge theories has shed new light on this.

2. MOTION OF A STRING

In the approximation that the gravitational field of the string is neglected, the motion of the string is described by the Nambu action (Vilenkin, 1985; Nambu, 1969, 1970; Nielsen and Olesen, 1973; Maeda and Turok, 1987):

$$I = -\mu \int d^2 l \left[-\det\left(g_{\alpha\beta} \frac{\partial x^\alpha}{\partial l^a} \frac{\partial x^\beta}{\partial l^b} \right) \right]^{1/2} \tag{1}$$

where μ is the mass of the string per unit length, $g_{\alpha\beta}$ (α, β = 1, 2, 3, 4) is an external gravitational field, and l^a denotes the world-sheet coordinates (a, b = 0, 1; $l^0 = \tau$, $l^1 = \sigma$). We consider the string to be open and infinite. In this case we also suppose that the force is applied to the string at infinity, so that the string will not fall to the source responsible for creating the spacetime concerned.

In general a stationary spacetime is given by

$$ds^2 = -R(dt + L_i\, dx^i)^2 + \frac{1}{R} l_{ij}\, dx^i\, dx^j \tag{2}$$

where $\partial_i R = \partial_i L_i = \partial_i l_{ij} = 0$ and i, j = 2, 3, 4. For time-independent string configurations where $\tau = t$ and the spacelike coordinates x^i depend on σ, the Nambu action can be written as

$$I = -\mu \int d\sigma \left(l_{ij} \frac{dx^i}{d\sigma} \frac{dx^j}{d\sigma} \right)^{1/2} \Delta t \tag{3}$$

Since the equilibrium configurations correspond to minimal energy, the problem is reduced to the investigation of the geodesics in three-dimensional space with the metric

$$ds^2 = l_{ij}\, dx^i\, dx^j \tag{4}$$

3. EQUILIBRIUM CONFIGURATION

For the Kerr–Newman–Kasuya spacetime we have

$$R = \frac{\Delta - a^2 \sin^2\vartheta}{r^2 + a^2 \cos^2\vartheta} \tag{5}$$

$$L_i = \delta_i^\varphi K_\varphi \tag{6}$$

where

$$K_\varphi = \frac{a \sin^2\vartheta\,(2Mr - e^2 - g^2)}{\Delta - a^2 \sin^2\vartheta} \qquad \text{and}$$

$$\Delta = r^2 - 2Mr + a^2 + e^2 + g^2$$

Here M, a, e, and g are the mass, angular momentum per unit mass, and electric and magnetic monopole charge parameters, respectively. The three-dimensional metric l_{ij} become

$$l_{ij} = 0, \qquad \text{where} \quad i \neq j \tag{7a}$$

$$l_{rr} = \frac{\Delta - a^2 \sin^2\vartheta}{\Delta} \tag{7b}$$

$$l_{\vartheta\vartheta} = \Delta - a^2 \sin^2\vartheta \tag{7c}$$

$$l_{\varphi\varphi} = \Delta \sin^2\vartheta \tag{7d}$$

For our study of the geodesics of the three-dimensional space l_{ij} we will use the Hamilton–Jacobi method (Misner *et al.*, 1973). We can write the Hamilton–Jacobi equation of the metric l_{ij} as

$$\frac{\partial S}{\partial \sigma} + \frac{1}{2}\, l^{ij} \frac{\partial S}{\partial x^i} \frac{\partial S}{\partial x^j} = 0 \tag{8}$$

where σ is an affine parameter along the geodesic.

If we write

$$S = -\tfrac{1}{2} q^2 \sigma + k\varphi + P(r) + Q(\vartheta) \tag{9}$$

then we have from (7)–(9)

$$\Delta\left(\frac{dP}{dr}\right)^2 - \frac{a^2 k^2}{\Delta} - q^2 \Delta = -m^2 \tag{10a}$$

$$\left(\frac{dQ}{d\vartheta}\right)^2 + \frac{k^2}{\sin^2\vartheta} + q^2 a^2 \sin^2\vartheta = m^2 \tag{10b}$$

where m^2 is the separation constant. The integral of motion k corresponds to the Killing vector $\eta_\varphi = \partial/\partial\varphi$ and m is related to the existence of the Killing tensor η_{ij}:

$$m^2 = \eta^{ij} p_i p_j \tag{11}$$

where

$$p_i = \frac{\partial S}{\partial x^j} \tag{12}$$

and

$$\eta_i^j = \text{diag}(a^2 \sin^2\vartheta,\ \Delta,\ \Delta + a^2 \sin^2\vartheta) \tag{13}$$

On integration from (10) we have

$$P(r) = \int^r dr\ \sqrt{H} \tag{14a}$$

$$Q(\vartheta) = \int^\vartheta d\vartheta\ \sqrt{\theta} \tag{14b}$$

where

$$H = \frac{a^2k^2}{\Delta^2} - \frac{m^2}{\Delta} + q^2 \tag{15}$$

and

$$\theta = m^2 - \frac{k^2}{\sin^2\vartheta} - q^2a^2 \sin^2\vartheta \tag{16}$$

Therefore equation (9) can be written as

$$S = -\frac{1}{2}\, q^2\sigma + k\varphi + \int^r \sqrt{H}\, dr + \int^\vartheta \sqrt{\theta}\, d\vartheta \tag{17}$$

By differentiating (17) with respect to q^2, m, and k and setting each of the derivatives equal to zero, we obtain the equations

$$\sigma - \sigma_0 = \int_{r_0}^r \frac{dr}{\sqrt{H}} - a^2 \int_{\vartheta_0}^\vartheta \frac{\sin^2\vartheta}{\sqrt{\theta}}\, d\vartheta \tag{18}$$

$$\int_{r_0}^r \frac{dr}{\Delta\sqrt{H}} = \int_{\vartheta_0}^\vartheta \frac{d\vartheta}{\sqrt{\theta}} \tag{19}$$

$$\varphi - \varphi_0 = k\left(\int_{\vartheta_0}^\vartheta \frac{d\vartheta}{\sin^2\vartheta\sqrt{\theta}} - a^2 \int_{r_0}^r \frac{dr}{\Delta^2\sqrt{H}}\right) \tag{20}$$

Equations (18)–(20) describe the equilibrium configuration of a string passing through the point (r_0, ϑ_0, φ_0) where the value of the affine parameter σ is σ_0. The string lies on a rotational surface given by equation (19). Equation (20) provides a unique curve on this surface. Since q is an inessential parameter, it can be changed by redefinition of the affine parameter σ. From now on we set $q = 1$.

Equations (18)–(20) can be put in the following form:

$$p_r^2 = \left(l_{rr} \frac{dr}{d\sigma}\right)^2 = H \tag{21}$$

$$p_\vartheta^2 = \left(l_{\vartheta\vartheta} \frac{d\vartheta}{d\sigma}\right)^2 = \theta \tag{22}$$

$$p_\varphi^2 = \left(l_{\varphi\varphi} \frac{d\varphi}{d\sigma}\right)^2 = k^2 \tag{23}$$

To analyze the form of the rotational surface, we rewrite (22) as

$$p_\vartheta^2 = m^2 - \frac{k^2}{\sin^2\vartheta} - a^2 \sin^2\vartheta - m^2 - V(\vartheta)$$

where

$$V(\vartheta) = \frac{k^2}{\sin^2\vartheta} + a^2 \sin^2\vartheta$$

If $m^2 = V(\vartheta)$ is the minimal value of the function $V(\vartheta)$ at $\vartheta = \vartheta_0$, the solution of (22) is $\vartheta = \vartheta_0$. In this particular case the surface on which the string lies is conelike.

When $k < a$, we get $\vartheta_0 = \arcsin(|k/a|^{1/2})$, $V(\vartheta_0) = 2a|k|$. In the case $k > a$ we get $\vartheta_0 = \pi/2$ and $V(\vartheta_0) = k^2 + a^2$. For $k^2 > a^2$ and $m^2 = k^2 + a^2$ the string lies in the equatorial plane of the source responsible for the Kerr–Newman–Kasuya spacetime.

4. DISCUSSION

In the case $g = 0$ the result obtained in this paper reduces to the result obtained by Frolov *et al.* (1989).

This study not only encompasses the result obtained by Frolov *et al.*, but also provides a similar result if the Kerr–Newman spacetime is involved with the magnetic monopole. So it is interesting to note that the physical results remain the same whether or not the magnetic monopole does exist in nature.

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